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ON THE PERTURBATIVE EQUIVALENCE BETWEEN

THE HAMILTONIAN AND LAGRANGIAN QUANTIZATIONS

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Abstract

The Hamiltonian (BFV) and Lagrangian (BV) quantization schemes are proved to be equivalent perturbatively to each other. It is shown in particular that the quantum master equation being treated perturbatively possesses a local formal solution.

1 Introduction

The Hamiltonian (BFV) [l-4] and Lagrangian (BV) [5 – 7] quantization schemes are the most popular ones presently. Both these approaches are shown to be equivalent to each other as applied to all the known field–theoretic examples (Yang-Mills theory, gravity, etc.). However it would be very interesting in principle to compare these approaches in the most general case.

It is relevant to elucidate here what we mean by the term "equivalence". As it is well–known [8], the general solution to generating equations of the BFV–method is determined up to a canonical transformation and a choice of gauge–fixing fermion, so that all such solutions are physically-equivalent. On the other hand, in the BV–method the general solution to the classical master equation is determined [10, 6] up to an anticanonical transformation, and all such solutions are also physically–equivalent. The general solution to the quantum master equation is determined [11, 12] up to an anticanonical transformation accompanied by adding gauge–invariant "quantum corrections" $\sim O(\hbar)$ to the initial classical action. These "quantum corrections" result in the fact that different solutions to the quantum master equation may appear to be physically–nonequivalent. Therefore, when reasoning on the physical equivalence between two different quantum master actions (or even between two different quantum theories at all) we mean the following: there exists a solution to the first quantum master equation (or an action of the first theory), which is physically–equivalent to the given solution to the second master equation (or to the given action of the second theory).

In principle, it has been shown in Ref. [13] that there exists an effective action depending only on the subset of phase variables of the BFV–quantization (in particular, on the variables of the Lagrangian BV–quantization only), as well as on the corresponding anti-variables, which effective action satisfies the quantum master equation. It remains, however, unclear what is the relation between the mentioned effective action and the initial one. The equivalence between the two approaches has been proved in Ref. [14, 15] for theories with first–class constraints only (and only up to a local measure as for Ref. [15]). As a consequence of the result, it was shown that the quantum master equation as applied to such theories possesses a local solution. In Ref. [16] the general case of theories with constraints of the both classes has been considered. However, the equivalence of the two approaches to the quantization problem was proved here up to a local measure. In the present paper we eliminate all the above mentioned restrictions and prove the formal perturbative equivalence between the Hamiltonian and Lagrangian quantizations as applied to an arbitrary classical theory. As a basic point, we generalize the result of Ref. [17] about the quantum equivalence of the classically–equivalent theories.

We treat of the classical equivalence in the following sense. Let the motion equations

$$\frac{\delta \mathcal{S}(x,y)}{\delta y} = 0$$

determine y uniquely to be some functions f of x: $y = f(x)$. Then the actions $\mathcal{S}(x,y)$ and $\mathcal{S}_1(x) \equiv \mathcal{S}(x, f(x))$ are classically-equivalent. Usually we also apply the same terminology directly to the theories with actions $\mathcal{S}$ and $\mathcal{S}_1$.

It was shown in Ref. [17] that, given a solution to the master equation for the theory

with classical action $\mathcal{S}_1(x)$, then there exists a physically–equivalent solution to the master equation for the theory with classical action $\mathcal{S}(x, y)$.

In Section 2 we extend the result of Ref. [17] to cover the case of quantum master equations and, what is more, we prove the coresponding vice–versa result as well.

In Section 3, given the classical action $\mathcal{S}_L$, we construct the classically–equivalent action $\mathcal{S}_{1H}$ to be convenient to us, which has the form of a Hamiltonian action with first–class constraints only.

In Section 4, by making use of the method of Ref.[14], we prove the existence of a solution to the master equation for the theory with classical action $\mathcal{S}_{1H}$, which solution is in fact equivalent to the Dirac quantization of the classical theory $\mathcal{S}_L$. It follows then from the results of Section 2 that there exists a solution to the quantum master equation for the theory with classical actions $\mathcal{S}_L$, which solution is equivalent to the Dirac quantization and, thereby, is equivalent to the BFV–one.

For the sake of simplicity, in what follows below we restrict ourselves by the case of purely bosonic initial Lagrangian variables.

2 Quantum equivalence of classically–equivalent theories in Lagrangian approach

Let us consider the action $\mathcal{S}(\varphi) \equiv \mathcal{S}(x, y)$ such that the equations of motion

$$\frac{\delta \mathcal{S}}{\delta y^a} = 0 \tag{2.1}$$

determine y^a uniquely to be some functions f^a of x:

$$\frac{\delta \mathcal{S}}{\delta y^a} = 0 \quad \leftrightarrow \quad y^a = f^a(x). \tag{2.2}$$

The reduced action $\mathcal{S}_1(x)$:

$$\mathcal{S}_1(x) \equiv \mathcal{S}(x, f(x)) \tag{2.3}$$

is physically–equivalent to the initial one $\mathcal{S}(x, y)$.

If one supposes the initial action $\mathcal{S}(x, y)$ to be gauge–invariant, with

$$R_\alpha^A(x, y), \quad \varphi^A \equiv (x^i, y^a), \tag{2.4}$$

being the corresponding gauge generators, then the reduced functions

$$R_{1\alpha}^i(x) \equiv R_\alpha^i(x, f(x)) \tag{2.5}$$

serve as gauge generators to the reduced action $\mathcal{S}_1(x)$, and vice versa as well.

Let us derive a property of the action $\mathcal{S}(x, y)$ to be important in what follows below. By making the following unimodular change of variables

$$\varphi^A \quad \rightarrow \quad \varphi'^A = (x^i, z^a = y^a - f^a(x)) \tag{2.6}$$

let us transform the action to be represented in terms of new variables as

$$\mathcal{S}(x,y) = \mathcal{S}'(x,z) = \mathcal{S}_1(x) + \frac{1}{2}\int dt z^a \Lambda_{ab}(x,z)z^b \tag{2.7}$$

with $\Lambda_{ab}(x,z)$ being finite–order differential operator (FODO) with coefficients depending on x, z.

Lemma 1. There exists a change of variables

$$z^a \quad \rightarrow \quad Y^a = \mu_b^a(x,z)z^b, \quad \mu_b^a(x,z) = \delta_b^a + O(\varphi'), \tag{2.8}$$

with $\mu_b^a(x,z)$ being FODO to each finite order in x, z, such that the action takes the form

$$\mathcal{S}'(x,z) = \mathcal{S}(x) + \frac{1}{2}\int dt Y^a \Lambda_{ab}^{(0)} Y^b \tag{2.9}$$

where

$$\Lambda_{ab}^{(0)} = \Lambda_{ab}(0,0). \tag{2.10}$$

Proof. Let us suppose that we have constructed FODO $\mu^{(n)a}{}_b(x,z)$ such that

$$\mathcal{S}'(x,z) = \mathcal{S}_1(x) + \frac{1}{2}\int dt[Y^{(n)a}\Lambda_{ab}^{(0)}Y^{(n)b} + z^a \delta_{n+1}\Lambda_{ab}(x,z)z^b] \tag{2.11}$$

where

$$Y^{(n)a} = \mu^{(n)a}{}_b(x,z)z^b \tag{2.12}$$

$$\delta_{n+1}\Lambda_{ab}(x,z) = O((\varphi')^{n+1}) \tag{2.13}$$

with $\delta_{n+1}\Lambda_{ab}$ being FODO. Let us choose

$$Y^{(n+1)a} = \left(\mu^{(n)a}{}_b(x,z) + \frac{1}{2}\overrightarrow{\Sigma}^{ac}\delta_{n+1}\overrightarrow{\Lambda}_{cb}(x,z)\right)z^b, \tag{2.14}$$

where, by definition, we set

$$\int dt \varphi^a Q_{ab}\psi^b = \int dt \varphi^a \psi_{Qa} = \int dt \varphi_{Qa}\psi^a, \tag{2.15}$$

$$\varphi_{Qa} \equiv \overrightarrow{Q}_{ab}\varphi^b, \tag{2.16}$$

for arbitrary Hermitean FODO Q_{ab}, and

$$\left(\overrightarrow{\Sigma}^{ab}\overrightarrow{\Lambda}_{bc}^{(0)}\gamma^c\right)(t) = \gamma^a(t) \tag{2.17}$$

for arbitrary function of time $\gamma^a(t)$.

We claim that $\overrightarrow{\Sigma}^{ab}$ is FODO. Indeed, to the lowest order in φ' the motion equation

$$\frac{\delta\mathcal{S}'}{\delta z^a} = 0 \tag{2.18}$$

takes the form

$$\overrightarrow{\Lambda}^{(0)}_{ab} z^b = 0 \tag{2.19}$$

and has the unique solution $z^a = 0$. This implies that $\det \overrightarrow{\Lambda}^{(0)}_{ab}$ is a number (not a function of the time–differentiation operators), and, hence, the matrix $\overrightarrow{\Lambda}^{(0)}_{ab}$ and its inverse as well are both FODO. Thus one can write down

$$Y^{(n+1)a} = \mu^{(n+1)a}{}_b(x,z) z^b \tag{2.20}$$

with $\mu^{(n+1)a}{}_b$ being FODO.

It is easy to check the action $\mathcal{S}'(x,z)$, in terms of $Y^{(n+1)a}$, to have the form

$$\mathcal{S}'(x,z) = \mathcal{S}_1(x) + \frac{1}{2} \int dt [Y^{(n+1)a} \Lambda^{(0)}_{ab} Y^{(n+1)b} + z^a \delta_{n+2} \Lambda_{ab}(x,z) z^b] \tag{2.21}$$

with $\delta_{(n+2)} \Lambda_{ab}$ of the (n+2)-th order in φ', being some FODO. As the assumption is obviously fulfilled to the lowest order in φ' ($n = 0$, $\mu^{(0)a}{}_b = \delta^a_b$), one should apply the induction method to complete the Proof.

The inverse of (2.6) has the form

$$z^a = \tilde{\mu}^a_b(x,Y) Y^b \tag{2.22}$$

with $\tilde{\mu}^a_b$ being some FODO. Notice that, formally, the functional determinant

$$\mathrm{Det} \frac{\delta \varphi'^A(t')}{\delta \varphi^B(t)} \tag{2.23}$$

can be represented in the form

$$\mathrm{Det} \frac{\delta \varphi'^A(t')}{\delta \varphi^B(t)} = \exp[\mathrm{Sp} \ln \frac{\delta Y^a(x(t'), z(t'))}{\delta z^b(t)}] = \exp[\delta(0) \int dt F(x,z)], \tag{2.24}$$

where $F(x,z)$ is a local functional of x, z and their time–derivatives (of a finite order to each finite order in φ).

Lemma 2. Given any solution to the quantum master equation for the theory with classical action $\mathcal{S}_1(x)$, then a solution, physically–equivalent to the given one, does exist to satisfy the quantum master equation for the theory with classical action $\mathcal{S}(x,z)$.

Proof. Let $\Phi^I_1 = (x^i, c^\alpha, \bar{c}_\alpha, B_\alpha)$ be the complete set of variables required to the Lagrangian BFV–quantization scheme, and let Φ^*_{1I} be the corresponding set of antivariables. In its own turn, let $W_1(\Phi_1, \Phi^*_1)$ be a solution to the quantum master equation,

$$\Delta_1 \exp[\frac{\imath}{\hbar} W_1] = 0, \tag{2.25}$$

$$\Delta_1 = (-1)^{\varepsilon(\Phi^I_1)} \int dt \frac{\delta}{\delta \Phi^I_1} \frac{\delta}{\delta \Phi^*_{1I}}. \tag{2.26}$$

For the theory with classical action $\mathcal{S}(x,z)$ we construct a solution, physically–equivalent to the one $W_1(\Phi_1, \Phi^*_1)$, in the following way.

We start from the theory, described in terms of the variables $\tilde{\Phi}^{\mathcal{A}} = (\tilde{\Phi}^I_1, \tilde{Y}^a)$, $\tilde{\Phi}^*_{\mathcal{A}} = (\tilde{\Phi}^*_{1I}, \tilde{Y}^*_a)$, with classical action $\tilde{\mathcal{S}}(\tilde{x}, \tilde{y}) = \mathcal{S}_1(\tilde{x}) + \frac{1}{2} \tilde{Y}^a \Lambda^{(0)}_{ab} \tilde{Y}^b$.

Next, we choose the solution

$$\tilde{W} = W_1(\tilde{\Phi}_1, \tilde{\Phi}_1^*) + \frac{1}{2}\tilde{Y}^a\Lambda_{ab}^{(0)}\tilde{Y}^b \qquad (2.27)$$

to satisfy the quantum master equation

$$\tilde{\Delta}\exp[\frac{\imath}{\hbar}\tilde{W}] = 0, \quad \tilde{\Delta} = \tilde{\Delta}_1 + \int dt\frac{\delta}{\delta\tilde{Y}^a}\frac{\delta}{\delta\tilde{Y}_a^*}, \qquad (2.28)$$

and boundary condition

$$\tilde{W}\Big|_{\tilde{\Phi}^*=0,\hbar=0} = \tilde{\mathcal{S}}. \qquad (2.29)$$

By making use of the anticanonical transformation with generating function

$$X = \int dt\left(\tilde{Y}_a^*Y^a(x,z) + \tilde{\Phi}_{1I}^*\Phi_1^I\right), \qquad (2.30)$$

let us pass then to the variables $\Phi^{\mathcal{A}} = (\Phi_1^I, z^a, \Phi_{1I}^*, z_a^*)$, which change takes the following explicit form

$$\Phi_1^I = \tilde{\Phi}_1^I, \quad \tilde{Y}^a = Y^a(x,z), \qquad (2.31)$$

$$\Phi_{1I}^* = \tilde{\Phi}_{1I}^*, \quad \text{for} \quad \Phi_{1I}^* \neq x_i^*, \qquad (2.32)$$

$$z_a^* = \overrightarrow{Y}_a^b\tilde{Y}_b^*, \quad \tilde{Y}_a^* = \overrightarrow{Z}_a^b z_b^*, \qquad (2.33)$$

$$\tilde{x}_i^* = x_i^* - \overrightarrow{Y}_i^a\overrightarrow{Z}_a^b z_b^*, \qquad (2.34)$$

with differential operators $\overrightarrow{Y}_b^a$, $\overrightarrow{Y}_i^a$, $\overrightarrow{Z}_b^a$ to be determined from the relations

$$\overrightarrow{Y}_b^a\tilde{Y}_a^*(t) \equiv \frac{\delta}{\delta z^a(t)}X, \quad \overrightarrow{Z}_b^a\overrightarrow{Y}_c^b = \delta_c^a, \qquad (2.35)$$

$$\overrightarrow{Y}_i^a\tilde{Y}_a^*(t) \equiv \frac{\delta}{\delta x^i(t)}X - \tilde{x}_i^*(t), \qquad (2.36)$$

which, in their own turn, imply all these operators to be, perturbatively, FODO. What is more, the change (2.31) - (2.34) and its inverse as well are both perturbatively–local. It has been shown in Refs. [11, 12] that given $W(\Phi, \Phi^*)$ (Φ is a complete set of variables) to satisfy the quantum master equation, then the action $W_X(\Phi, \Phi^*)$:

$$W_X(\Phi, \Phi^*) \equiv W(\Phi_X, \Phi_X^*) - \imath\hbar\Delta_X, \qquad (2.37)$$

$$\Delta_X = \frac{1}{2}\ln\text{SDet}\frac{\delta(\Phi_X, \Phi_X^*)}{\delta(\Phi, \Phi^*)} = \ln\text{SDet}\frac{\delta\Phi_X}{\delta\Phi}, \qquad (2.38)$$

with Φ_X, Φ_X^* being an anticanonical transform of Φ, Φ^*, does the same.

In the case under consideration we choose (in terms of the variables $\Phi = (\Phi_1, z)$)

$$W(\Phi, \Phi^*) \equiv \tilde{W}\left(\tilde{\Phi}(\Phi, \Phi^*), \tilde{\Phi}^*(\Phi, \Phi^*)\right) - \imath\hbar \mathrm{Det}\frac{\delta\tilde{\Phi}^{\mathcal{A}}(\Phi, \Phi^*)}{\delta\Phi^{\mathcal{A}'}}. \tag{2.39}$$

to be a "particular" solution to the quantum master equation

$$\Delta \exp[\frac{\imath}{\hbar}W] = 0, \quad \Delta = \Delta_1 + \int dt \frac{\delta}{\delta z^a}\frac{\delta}{\delta z_a^*}. \tag{2.40}$$

At the same time the solution chosen obviously satisfies the boundary condition

$$W\Big|_{\Phi^*=0,\hbar=0} = \mathcal{S}(x, z). \tag{2.41}$$

Let us check the actions $W_1(\Phi_1, \Phi_1^*)$ and $W(\Phi, \Phi^*)$ to be physically–equivalent. In the theory, parametrized by the mentioned variables Φ, Φ^* let us choose the gauge fermion Ψ to depend on the variables Φ_1 only:

$$\Psi = \Psi(\Phi_1). \tag{2.42}$$

Then we have for the statsum Z:

$$Z = \int D\Phi \exp[\frac{\imath}{\hbar}W\Big|_{\Phi_1^*=\frac{\partial\Psi}{\partial\Phi_1}, z^*=0}] = \int D\Phi \exp[\frac{\imath}{\hbar}W_1\Big|_{\Phi_1^*=\frac{\partial\Psi}{\partial\Phi_1}}] = Z_1. \tag{2.43}$$

It can be also easily checked that, to the first order in z^*, the expression (2.30), taken at $\hbar = 0$, coincides with the action given in Ref. [17].

The inverse of the Lemma 2 is valid too.

Lemma 3. Given any solution $W(\Phi, \Phi^*)$ to the quantum master equation (2.40), then a physically–equivalent solution $W_1(\Phi_1, \Phi_1^*)$ does exist to satisfy the quantum master equation (2.35), (2.36).

Proof. By choosing the gauge fermion Ψ to depend on Φ_1 only, we have

$$\exp[\frac{\imath}{\hbar}W_1(\Phi_1, \Phi_1^*)] \equiv \int Dz \exp[\frac{\imath}{\hbar}W\Big|_{z^*=0}] = \int DY \exp[\frac{\imath}{\hbar}\bar{W}], \tag{2.44}$$

$$\bar{W} = W\Big|_{z^*=0} + \imath\hbar \mathrm{Det}\frac{\delta Y^a}{\delta z^b} = \mathcal{S}_1(x) + \frac{1}{2}Y^a\Lambda_{ab}^{(0)}Y^b + M, \tag{2.45}$$

where $M = 0$ at $\Phi_1^* = 0$, $\hbar = 0$.

Obviously, W_1 is, formally, a local functional of Φ_1, Φ_1^* , and

$$W_1\Big|_{\Phi_1^*=0,\hbar=0} = \mathcal{S}_1(x). \tag{2.46}$$

Besides, by making use of the method of Ref. [13], one can check W_1 thus defined to satisfy the quantum master equation. We conclude the Section with the following claim: if the actions $\mathcal{S}$ and $\mathcal{S}_1$ are classically–equivalent in the sense of the relations (2.2), (2.3), then the corresponding BV–quantized theories are also physically–equivalent in the sense that the existence of a solution to the quantum master equation in the first theory implies the existence of a physically–equivalent solution to the quantum master equation in the second theory, and vice versa as well.

3 Proper Hamiltonian action classically–equivalent to Lagrangian one

Let us begin here with the Lagrangian action

$$\mathcal{S}_L \equiv \int dt L(q, \dot{q}). \tag{3.1}$$

Without a loss of generality one can suppose the Lagrangian L to depend only on coordinates and velocities.

Next, let us introduce [18] the action $\mathcal{S}_v$:

$$\mathcal{S}_v = \int dt(L(q, v) + p(\dot{q} - v)). \tag{3.2}$$

The action $\mathcal{S}_v$ is classically–equivalent to the one $\mathcal{S}_L$, as one obtains $\mathcal{S}_L$ by substituting $v \rightarrow v(q, \dot{q}) = \dot{q}$, $p \rightarrow p(q, \dot{q}) = \partial L/\partial \dot{q}$ into $\mathcal{S}_v$, which substitution, in its own turn, is determined by the equations

$$\frac{\delta \mathcal{S}_v}{\delta p} = \frac{\delta \mathcal{S}_v}{\delta v} = 0. \tag{3.3}$$

Let the velocities v, as well as the momenta p, are split into the corresponding subsets

$$v = (V, \lambda), \quad p = (\Pi, \pi) \tag{3.4}$$

in such a way that the submatrix

$$\frac{\partial^2 L}{\partial V \partial V} \tag{3.5}$$

is the maximal–size square block of the Hessian

$$\frac{\partial^2 L}{\partial v \partial v} \tag{3.6}$$

whereas the momenta Π are determined from the equations

$$\frac{\delta \mathcal{S}_v}{\delta V} = \frac{\partial L}{\partial V} - \Pi = 0. \tag{3.7}$$

These equations are solvable with respect to V's:

$$(3.7) \quad \Rightarrow \quad V = \bar{V}(\Pi, \lambda, q). \tag{3.8}$$

By substituting these V's into the action $\mathcal{S}_v$ one obtains the action $\mathcal{S}_H$ classically–equivalent to the action $\mathcal{S}_v$ (and hence to $\mathcal{S}_L$):

$$\mathcal{S}_H = \mathcal{S}_v\big|_{V \rightarrow \bar{V}} = \int dt(p\dot{q} - H(\Pi, q) - \lambda \Phi^{(1)}), \tag{3.9}$$

with $\Phi^{(1)} = \pi - f(\Pi, q)$ being primary constraints. It has been shown in Ref.[18] that there exists a point–like change of variables $(p, q, \lambda) \rightarrow (p', q', \lambda')$, which is a canonical

transformation as applied to the variables p, q, such that the action $\mathcal{S}_H$ takes the form (we omit primes)

$$\mathcal{S}_H \quad \rightarrow \quad \mathcal{S}'_H = \int dt \left(p\dot{q} - H_{ph}(\omega) - A(\omega, \mathsf{P}^{(2)}, Q)\mathsf{P}^{(2)} - \Delta H - \lambda_\mathsf{P}\mathsf{P}^{(1)} - \lambda_\theta\theta^{(1)} \right) \quad (3.10)$$

where $\eta' = (p', q') = (\omega; \mathsf{P}, Q; \theta)$, θ is a set of canonical pairs describing second–class constraints, P is a set of momenta describing first–class constraints, Q is a set of coordinates canonically–conjugated to P, ω is a set of canonical pairs describing physical degrees of freedom, $\theta^{(1)}$ is a set of primary second–class constraints, $\mathsf{P}^{(1)}$ is a set of primary first-class constraints, λ_P and λ_θ are Lagrange multipliers, $\theta^{(2)}$ represent all secondary second–class constraints (the ones of the second, third, etc. steps of the Dirac's procedure), $\mathsf{P}^{(2)}$ represent all secondary first-class constraints. The function ΔH has the structure

$$\Delta H = O\left(\mathsf{P}^{(2)}\theta^{(2)}, (\theta^{(2)})^2 \right) \quad (3.11)$$

and can be canonically transformed to become

$$\Delta H = \frac{1}{2}\theta^{(2)}b\theta^{(2)} + \Delta_3 H \quad (3.12)$$

where $\Delta_3 H$ is of the third order in η'. Let us denote $\bar{\theta} = (\theta, \lambda_\theta)$. The equations of motion coming from $\mathcal{S}'_H$ by varying with respect to $\bar{\theta}$ have the structure

$$B\bar{\theta} = \frac{\partial \Delta_3 H}{\partial \bar{\theta}} \quad (3.13)$$

where B is a matrix differential (in time) operator entering a quadratic theory. As $\bar{\theta} = 0$ is the only solution in a quadratic theory, $\det B$ is a number, and B^{-1} is FODO. The eq. (3.13) determines $\bar{\theta}$ to be some functions of ω, P, Q and their time-derivatives of a finite order, to each finite order in P. In other words, $\bar{\theta}$ are perturbatively–local functionals (i.e. functions) of ω, P, Q and their time–derivatives, such that $\bar{\theta}\big|_{\mathsf{P}=0} = 0$.

Let us substitute the expressions obtained for $\bar{\theta}$ into $\mathcal{S}'_H$. As a result, one gets the action $\mathcal{S}''_H$ classically–equivalent to $\mathcal{S}'_H$ (and, hence, to $\mathcal{S}_L$), which has the structure

$$\mathcal{S}''_H = \int dt \left(\omega_p\dot{\omega}_q + \mathsf{P}\dot{Q} - H_{ph}(\omega) - A(\omega, \mathsf{P}^{(2)}, Q)\mathsf{P}^{(2)} - \lambda_\mathsf{P}\mathsf{P}^{(1)} + C \right), \quad (3.14)$$

where C is a local functional of ω, P, Q and their time–derivatives, which is at least quadratic in P.

The action $\mathcal{S}''$ does not depend on the variables θ, λ_θ which correspond to the second–class constraints. This action, however, is not of the Hamiltonian form, since the function C may depend on the time–differentiated phase variables. We are going to show that there exists a perturbatively–local change of variables that eliminates the function C.

For the sake of simplicity we restrict ourselves by a nonprincipal extra assumption that the same number of first-class constraints appears at each step of the Dirac procedure, which number equals to the one of primary first–class constraints. Thus we suppose that

$$\mathsf{P} = (\mathsf{P}_{(1)}, \mathsf{P}_{(2)}, \dots, \mathsf{P}_{(L)}), \quad \mathsf{P}_{(1)} \equiv \mathsf{P}^{(1)}. \quad (3.15)$$

One can show the contribution $A\mathsf{P}^{(2)}$ to take the form

$$A(\omega, \mathsf{P}^{(2)}, Q)\mathsf{P}^{(2)} = A(\omega, \mathsf{P}^{(2)}, 0)\mathsf{P}^{(2)} + \sum_{i=2}^{L}\left(Q_{(i-1)} + \sum_{k=1}^{L} Q_{(k)} a^{(k)(i)}(\omega)\right)\mathsf{P}_{(i)} +$$
$$+ O(Q(\mathsf{P}^{(2)})^2, Q^2 \mathsf{P}^{(2)}), \quad a^{(k)(i)}(\omega) = O(\omega). \tag{3.16}$$

as a result of a (linear) canonical transformation.

Now let us suppose the change

$$\bar{Q}_{(i)} \to \bar{Q}'_{(i)}, \quad Q_{(L)} \to Q'_{(L)} = Q_{(L)}, \quad \mathsf{P} \to \mathsf{P}' = \mathsf{P}, \quad \omega \to \omega' = \omega, \tag{3.17}$$

where

$$\bar{Q}_{(1)} = \lambda_\mathsf{P}, \quad \bar{Q}_{(i)} = Q_{(i-1)}, \quad i = 2, \dots, L, \tag{3.18}$$

to make the action $\mathcal{S}''_H$ take the form (3.14) with the function C being of the minimal order $n \geq 2$ in P. By integrating over time by part one can free one of the factors P from the time–differentiation, so that the C–contribution takes the form

$$\int dt \left(\mathsf{P}_{(i)} T^{(i)} + O(\mathsf{P}^{n+1})\right), \quad T^{(i)} = O(\mathsf{P}^{n-1}). \tag{3.19}$$

Next, let us make the change

$$\bar{Q}_{(i)} \quad \to \bar{Q}'_{(i)} = \bar{Q}_{(i)} + \Delta_{(i)}, \tag{3.20}$$

where $\Delta_{(i)}$ are determined by the equations

$$A_1^{(i)(j)} \Delta_{(j)} + A_2^{(i)(j)}(\omega) \Delta_{(j)} = T^{(i)}, \tag{3.21}$$

$$A_1^{(i)(j)} = \delta_{ij} - \delta_{i,j-1} \frac{d}{dt}, \tag{3.22}$$

$$A_2^{(1)(j)}(\omega) = A_2^{(i)(1)}(\omega) = 0; \quad A_2^{(i)(j)}(\omega) = a^{(i)(j-1)}(\omega), \quad i, j \geq 2. \tag{3.23}$$

As the operator A_1^{-1} is FODO, the operator $(A_1 + A_2)^{-1}$ does perturbatively the same, and $\Delta_{(j)}$ are local functionals of ω, P, Q and their time–derivatives of finite orders.

As a result of such a change the action $\mathcal{S}''_H$ takes the form (3.14) with the function C being of the order $\sim \mathsf{P}^{n+1}$.

Finally we conclude the initial action $\mathcal{S}_L$ to be dynamically–equivalent to the one $\mathcal{S}_{1H}$ (up to an invertible local in time change of variables):

$$\mathcal{S}_{1H} = \mathcal{S}_H\big|_{\theta=\lambda_\theta=0} = \int dt(\omega_p \dot{\omega}_q + \mathsf{P}\dot{Q} - H_{ph}(\omega) - A\mathsf{P}^{(2)} - \lambda_\mathsf{P} \mathsf{P}^{(1)}). \tag{3.24}$$

The action $\mathcal{S}_{1H}$ is of the Hamiltonian form, and generates first-class constraints only.

4   Solving quantum master equation

In this Section we construct the solution W_1 to the quantum master equation for the theory with classical action $\mathcal{S}_{1H}$ (3.24), which solution is physically–equivalent to the Dirac quantization of the theory (3.24). As the Dirac method applied to the theory (3.24) yields the answer

$$Z = \int D\omega \exp[\frac{\imath}{\hbar} \int dt(\omega_p \dot{\omega}_q - H_{ph}(\omega)) \tag{4.1}$$

which coincides with the Dirac quantization of the initial theory (3.1), we thereby guarantee that the quantum theory with the action W_1 is physically–equivalent to the theory quantized by applying the Dirac method and, hence, to the BV–quantized theory.

To construct W_1 we make use of the scheme suggested in Ref. [14]. Let us split the complete set of phase variables Γ of the BFV–quantization applied to the theory (3.24),

$$\Gamma = \left(\omega; \quad \mathsf{P}_{(i)}, Q_{(i)}; \quad C_{(i)}, \bar{\mathcal{P}}_{(i)}; \quad \bar{C}_{(i)}, \mathcal{P}_{(i)}; \quad \lambda_{(i)}, \pi_{(i)}\right), \quad i = 1, \ldots, L, \tag{4.2}$$

into the two groups x and y, where

$$x = (\omega, \quad \mathsf{P}, \quad Q, \quad \lambda \equiv \lambda_{(1)} \equiv \lambda_{\mathsf{P}}, \quad C \equiv C_{(L)}, \quad \bar{C} \equiv \bar{C}_{(L)}, \quad B \equiv \pi_{(L)}), \tag{4.3}$$

and the rest of the variables is included into the group y.

Let us define the unitarizing Hamiltonian via the formula

$$H = \mathcal{H}_{min}(\omega, \mathsf{P}, Q, C_{(i)}, \bar{\mathcal{P}}_{(i)}) + \{\Psi, \Omega\}, \tag{4.4}$$

where

$$\Omega = \Omega_{min}(\omega, \mathsf{P}, Q, C_{(i)}, \bar{\mathcal{P}}_{(i)}) + \pi_{(i)} \mathcal{P}_{(i)}, \tag{4.5}$$

the Fermionic Ω_{min} and Bosonic $\mathcal{H}_{min}$ generating functions of the constraint algebra satisfy the generating equations

$$\{\Omega_{min}, \Omega_{min}\} = 0, \quad \Omega_{min} = \mathsf{P}_{(i)} C_{(i)} + O(\bar{\mathcal{P}} C^2), \tag{4.6}$$

$$\{\mathcal{H}_{min}, \Omega_{min}\} = 0, \quad \mathcal{H}_{min} = H_{ph}(\omega) + \sum_{i=2}^{L} A(\omega, \mathsf{P}^{(2)}, Q)^{(i)} \mathsf{P}_{(i)} + \bar{\mathcal{P}}_{(i)} V^{(i)(j)} C_{(j)} + O(C^2 \bar{\mathcal{P}}^2), \tag{4.7}$$

and

$$V^{(i)(j)} = \frac{\partial A^{(i)}}{\partial Q_{(j)}} \tag{4.8}$$

Let us choose the gauge Fermion Ψ to have the form

$$\Psi = -\Psi_s(x) + \Psi_1(y) \tag{4.9}$$

where

$$\Psi_1(y) = \bar{\mathcal{P}}_{(1)}\lambda + b^2 \sum_{i=1}^{L-1} \bar{C}_{(i)}(\pi_{(i)} + \lambda_{(i+1)}). \tag{4.10}$$

Then let us define the action $\tilde{W}_b(x, x^*)$ to be

$$\exp[\frac{\imath}{\hbar}\tilde{W}_b(x, x^*)] = \int Dy \exp[\frac{\imath}{\hbar}\int dt(\Gamma_p\dot{\Gamma}_q - \tilde{H})] \tag{4.11}$$

where

$$\tilde{H} = H - x^*\{x, \Omega\} = \mathcal{H}_{min} + \{\Psi_1, \Omega\} - (x^* + \frac{\partial\Psi_s}{\partial x})\{x, \Omega\}. \tag{4.12}$$

The action $\tilde{W}_b$ has the two important properties:
   i) $\tilde{W}_b$ satisfies [13] the quantum master equation

$$\Delta_x \exp[\frac{\imath}{\hbar}\tilde{W}_b] = 0, \quad \Delta_x = (-1)^{\varepsilon(x)}\int dt\frac{\delta}{\delta x}\frac{\delta}{\delta x^*}; \tag{4.13}$$

   ii) x^* and Ψ_s enter $\tilde{W}_b$ only through the combination $\bar{x}^* \equiv x^* + \partial\Psi_s/\partial x$:

$$\tilde{W}_b(x, x^*) = W_b(x, \bar{x}^*). \tag{4.14}$$

It is thus sufficient for our purposes to analyze the action $W_b(x, x^*)$:

$$\exp[\frac{\imath}{\hbar}W_b(x, x^*)] = \int Dy \exp[\frac{\imath}{\hbar}\int dt(\Gamma_p\dot{\Gamma}_q - \mathcal{H}_{min} - \{\Psi_1, \Omega\} + x^*\{x, \Omega\})]. \tag{4.15}$$

Let us make the unimodular change of the integration variables

$$\lambda_{(i)} \quad \rightarrow \quad \frac{1}{b}\lambda_{(i)}, \quad \mathcal{P}_{(i)} \quad \rightarrow \quad \frac{1}{b}\mathcal{P}_{(i)}, \quad i \geq 2, \tag{4.16}$$

$$\pi_{(i)} \quad \rightarrow \quad \frac{1}{b}\pi_{(i)}, \quad \bar{C}_{(i)} \quad \rightarrow \quad \frac{1}{b}\bar{C}_{(i)}, \quad i \leq L-1, \tag{4.17}$$

and then consider the limit $b \to \infty$. We have

$$\exp[\frac{\imath}{\hbar}W_\infty] =$$

$$= \exp[\frac{\imath}{\hbar}(\mathcal{S}_{1H} + \int dtC^*B)]\int \prod_{i\geq 2} D\bar{\mathcal{P}}_{(i)} \prod_{k\leq L-1} DC_{(k)} \exp[\frac{\imath}{\hbar}\int dt(\bar{\mathcal{P}}_{(L)}\dot{C} + \sum_{i=2}^{L-1} \bar{\mathcal{P}}_{(i)}\dot{C}_{(i)} -$$

$$- \sum_{i=2}^{L}\sum_{j=1}^{L-1} \bar{\mathcal{P}}_{(i)}V^{(i)(j)}C_{(j)} + O(\lambda C_{(i)}\bar{\mathcal{P}}_{(j)}, C_{(i)}C_{(j)}\bar{\mathcal{P}}_{(k)}\bar{\mathcal{P}}_{(l)}) + x^*_{min}\{x_{min}, \Omega_{min}\}_0)], \tag{4.18}$$

where

$$x_{min} = (\omega; \mathsf{P}, Q; \lambda; C), \tag{4.19}$$

$$\{x_{min}, \Omega_{min}\}_0 \equiv \{x_{min}, \Omega_{min}\}\Big|_{\bar{\mathcal{P}}_{(1)}=0} \tag{4.20}$$

It follows from (4.18) that W_∞ has the structure

$$W_\infty(x, x^*) = W_{min}(x_{min}, x^*_{min}) + \int dt C^* B, \tag{4.21}$$

where W_{min}, obviously, satisfies the quantum master equation in the variables x_{min}, x^*_{min}, together with the boundary condition

$$W_{min}\Big|_{x^*_{min}=0,\hbar=0} = \mathcal{S}_{1H}. \tag{4.22}$$

Thus, given the classical action $\mathcal{S}_{1H}$, the action $\tilde{W}_\infty$ is constructed by the standard rules of the Lagrangian BV–quantization, so that the theory thus defined is physically–equivalent to the theory (3.1) quantized by applying the Dirac method. It remains to check the action W_{min} to be local.

Let us represent a quadratic in $\bar{\mathcal{P}}$, C contribution to the action in the exponent of the integrand of eq. (4.18) in the form (see also (3.16)):

$$\int dt \Big(\sum_{i=2}^{L-1} \bar{\mathcal{P}}_{(i)}\dot{C}_{(i)} - \sum_{i=2}^{L} \bar{\mathcal{P}}_{(i)}C_{(i-1)}\Big) \equiv \int dt C_{(i)}\sigma^{(i)(j)}\bar{\mathcal{P}}_{(j)}, \quad i \geq 2, j \leq L-1. \tag{4.23}$$

Notice that, to the linear order in the variables, the equations of motion, which follow from the action $\mathcal{S}_{1H}$ by varying with respect to $\bar{Q}_{(i)}$, have the form

$$\sum_{j=1}^{L} A^{(i)(j)}\mathsf{P}_{(j)} = 0, \quad A^{(i)(j)} = \begin{pmatrix} 1 & 0 \\ * & \sigma \end{pmatrix}. \tag{4.24}$$

As $\mathsf{P} = 0$ is the only solution to the equations (4.25), the time–differentiation operators do not enter $\det A = \det \sigma$, so that σ^{-1} is FODO (this follows also from the explicit form of the operator σ) and, hence, the propagators of the fields $C_{(i)}$, $i \leq L-1$, and $\mathsf{P}_{(j)}$, $j \geq 2$ are operators local in time.

So we have finally shown that there exists a local solution to the quantum master equation for the theory with classical action $\mathcal{S}_{1H}$, which solution is physically–equivalent to the quantization based on the Dirac formalism. As the initial action $\mathcal{S}_L$ is classically–equivalent to the one $\mathcal{S}_{1H}$, we conclude that a solution to the quantum master equation does exist for the initial theory too, which solution is physically–equivalent to the Dirac scheme. In that sense the Hamiltonian and Lagrangian quantizations are equivalent. Besides, it follows from the above consideration that, in general, the quantum master equation has, perturbatively, a local solution to satisfy a given boundary condition as well.

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